NON-LINEAR MECHANISM OF TSUNAMI GENERATION IN A COMPRESSIBLE OCEAN

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ANNOTATION

A nonlinear mechanism of long gravitational wave generation by bottom displacements (earthquakes) in a compressible water layer of constant depth is investigated analytically and numerically. The amplitude of the wave generated by the nonlinear mechanism is estimated as a function of ocean depth and of duration and velocity of bottom displacement. It is shown that this mechanism can provide a noticeable contribution to tsunami amplitude.

INTRODUCTION

Traditional view on tsunami generation mechanism is usually linked to a sudden displacement of water by residual bottom deformations during strong underwater earthquakes. Mathematical description of the tsunami generation often assumes that the bottom deformation is an instant process. At first sight, such approach is quite adequate because the duration τ of bottom deformation ($10^0 < \tau < 10^2$ sec) is always less than the time during which a long gravitational wave propagates over a tsunami source length scale L ($L(gH)^{-1/2} \sim 10^3$ sec, where H is ocean depth, g is acceleration due to gravity). Nevertheless, during rapid bottom movements ($\tau < 4Hc^{-1} \sim 10$ sec, where c is sound velocity in water) the water layer behaves as a compressible fluid [Nosov, 1999], and therefore the assumption of instantaneous bottom deformation is no longer acceptable.

The main response of a compressible water layer from bottom deformation are elastic oscillations, which are caused by repeated reflections of an acoustic impulse from bottom and top surfaces. Tsunami generation mechanism linked to such "rectification" of acoustic waves in ocean was reported for the first time in Novikova and Ostrovsky [1982].

The non-linear mechanism of generation of gravitational waves due to bottom oscillations in incompressible ocean was initially described in Nosov and Skachko [2001]. The present study is aimed to estimate the contribution of non-linear effects to tsunami amplitude in a case when the wave is generated in a compressible ocean by bottom deformations.

BASIC MATHEMATICAL MODEL

The mathematical model is based on the non-linear hydrodynamic equations

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}, \vec{\nabla}\right)\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \vec{g}\,, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho\, \vec{v}\right) = 0\,. \tag{2}$$

It is assumed that fluid velocity consists of variable (fast) and time averaged (slow) terms

$$\vec{v} = \langle \vec{v} \rangle + \vec{v}', \quad p = \langle p \rangle + p', \quad \rho = \langle \rho \rangle + \rho'. \qquad (3)$$

Substituting expressions (3) into equations (1) and (2) and averaging these equations in time we obtained

$$\frac{\partial \langle \vec{v} \rangle}{\partial t} = -\frac{\vec{\nabla}\langle p \rangle}{\langle \rho \rangle} + \vec{g} + \vec{f}, \qquad (4)$$

$$\mathrm{div}(\langle \vec{v} \rangle) = s, \qquad (5)$$

$$\vec{f} = -\left\langle \left(\vec{v}', \vec{\nabla}\right)\vec{v}' \right\rangle + \frac{\left\langle \vec{\nabla} p'^2 \right\rangle}{2c^2 \langle \rho \rangle^2}, \qquad (6)$$

$$s = -\frac{1}{\langle \rho \rangle}\mathrm{div}\langle \rho' \vec{v}' \rangle. \qquad (7)$$

When deriving equations (4) and (5) we neglected the non-linear term $(\langle \vec{v} \rangle, \nabla)\langle \vec{v} \rangle$ and assumed that the averaged fluid motion as incompressible. The non-linearity of equations (1) and (2) introduces additional terms in the time-averaged flow equations (4) and (5). These additional terms "$\vec{f}$" and "s" can be interpreted as external mass force and distributed mass source. In what follows, we consider "$\vec{f}$" and "s" as a *non-linear tsunami source*.

In order to calculate fields "$\vec{f}$" and "s", given by (6) and (7), the following functions need to be determined: $\vec{v}'$, p', and ρ'. Let us obtain these functions from a solution of an auxiliary linear problem of a linear response of compressible fluid to bottom deformations.

AUXILIARY LINEAR PROBLEM

Let us consider an ideal compressible homogeneous fluid layer of constant depth *H*. The origin of the Cartesian coordinate system *OXZ* is on the unperturbed free surface, with the *OZ*-axis oriented vertically upward. Small amplitude vertical bottom displacements $\eta(x,t)$ act as the wave source. This problem is solved using velocity potential F of the fluid:

$$\frac{\partial^2 F}{\partial t^2} - c^2 \Delta F = 0, \qquad (8)$$

$$\frac{\partial^2 F}{\partial t^2} = -g\frac{\partial F}{\partial z}, \quad z = 0, \qquad (9)$$

$$\frac{\partial F}{\partial z} = \frac{\partial \eta}{\partial t}, \quad z = -H, \qquad (10)$$

and the unknown functions are expressed in terms the velocity potential:

$$p' = -\rho_0 \frac{\partial F}{\partial t},$$
$$\rho' = c^{-2} p',$$
$$\vec{v}' = \vec{\nabla} F \qquad (11)$$

We have assumed a separable function for bottom displacements: $\eta(x,t) = \eta(x)\eta(t)$. In all cases the spatial distribution of bottom displacements $\eta(x)$ remains the same, whereas two different time-histories $\eta(t)$ were used: "piston" (bottom motions with residual displacements) and "membrane" (without residual displacements) were used. The corresponding functions $\eta(x)$ and $\eta(t)$ are shown on Figure 1.

Equations (8) – (10) were solved numerically using a finite-difference method [Nosov and Kolesov, 2002].

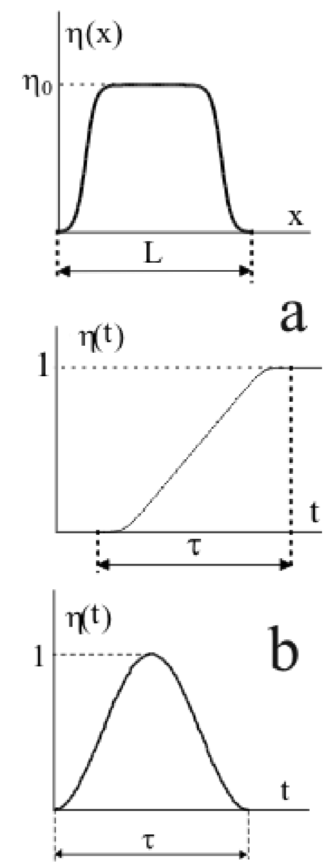

Figure 1. Time-spatial history of bottom displacements: (a) "piston" bottom displacements, (b) "membrane" bottom displacements.

GENERATION OF LONG GRAVITATIONAL WAVES BY A NON-LINEAR SOURCE

Generation of gravitational waves by joint action of the external mass force $\vec{f} = \{f_x, f_z\}$ and the distributed mass source was described using the framework of the long-wave linear theory. Neglecting vertical acceleration and integrating the equations (4) and (5) along the vertical coordinate from bottom up to free surface, we get the wave equation in terms of free surface displacements ξ:

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{gH}\frac{\partial^2 \xi}{\partial t^2} = \frac{1}{gH} Q(x,t), \qquad (12)$$

where

$$Q(x,t) = \int_{-H}^{0} dz \left[\frac{\partial f_x}{\partial x} + \int_{z}^{0} \frac{\partial^2 f_z}{\partial x^2} dz^* - \frac{\partial s}{\partial t} \right]. \qquad (13)$$

NORMAL MODE STRUCTURE OF WATER LAYER ELASTIC OSCILLATIONS OF AND NON-LINEAR TSUNAMI SOURCE

Certain conclusions about the properties of fields $\vec{f}$ and s and their contributions to the generation of long waves can be reached analytically as following. Elastic oscillations of a water layer exhibit a normal mode structure and all the unknown functions u', w', p' and ρ' depend on the fluid velocity potential.

In the beginning, let us consider a rather simple case of acoustic waves with vertical wave vectors. This corresponds to a "central" area of the tsunami source if the ocean bottom remains horizontal and moves uniformly in a vertical direction. This case degenerates into one-dimensional (along vertical axis Oz) motion. The considered domain is bounded above by the free surface and below by the absolutely rigid bottom. Therefore, any elastic movements in this domain can be represented as a superposition of normal oscillations with a discrete spectrum

$$F(x,y,z,t) = \sum_j F_0^j \cos(\omega_j t) \sin(n_j z), \qquad (14)$$

$$n_j = \frac{\pi}{2H}(1+2j)\,,\ j = 0,1,2,3...,\tag{15}$$

$$\omega_j = \frac{c\pi}{2H}(1+2j)\,,\tag{16}$$

where n_j is a vertical component of the wave vector and ω_j is its normal frequency.

Substituting in formula (6) u', w', p' и ρ', expressed in terms of the fluid velocity potential (14) we obtain

$$\begin{aligned} f_x &= 0, \\ f_z &= \frac{1}{2c^2}\sum_{jk} \omega_j\omega_k n_k F_0^j F_0^k \left[\cos\left((\omega_j-\omega_k)t\right) - \cos\left((\omega_j+\omega_k)t\right)\right]\sin(n_j z)\cos(n_k z) + \\ &+ \frac{1}{2}\sum_{jk} n_j n_k{}^2 F_0^j F_0^k \left[\cos\left((\omega_j-\omega_k)t\right) + \cos\left((\omega_j+\omega_k)t\right)\right]\cos(n_j z)\sin(n_k z). \end{aligned}\tag{17}$$

Terms of the series (17) with indexes j and k oscillate in time with frequencies $\omega_j - \omega_k = c\pi H^{-1}(j-k)$ or $\omega_j + \omega_k = c\pi H^{-1}(1+j+k)$. It should be noted, that expressions (17) need to be averaged over a period $T_0 = 2\pi/\omega_0$ and all the terms with $j \neq k$ vanish. This allows us to do the summation in (17) over a single index. Finally, we obtain

$$f_z = \frac{1}{2}\sum_j n_j{}^3 F_0^{j2}\sin(2n_j z)\tag{18}$$

Equation (18) shows that for j=0, the vertical component of the force is negative ($-H > z > 0$). As for the modes $j > 0$, the function f_z alternates in sign. But, when integrated along vertical coordinate, f_z contributes a long wave (formula (13)) as a force directed downward. The value of the integral is negative, independently on the index j.

In order to calculate the mass sources s, we considered the vector under the divergence operator in (7). The horizontal component of the vector is obviously zero. The vertical component is expressed through velocity potential:

$$\chi \equiv \frac{[p'\vec{v}']_z}{\langle\rho\rangle} = -\frac{\partial F}{\partial t}\frac{\partial F}{\partial z}.\tag{19}$$

Substituting the fluid velocity potential (14) in expression (19), we obtain

$$\begin{aligned} \chi &= \sum_{jk} \omega_j n_k F_0^j F_0^k \quad \sin(\omega_j t)\cos(\omega_k t)\sin(n_j z)\cos(n_k z) = \\ &= \frac{1}{2}\sum_{jk} \omega_j n_k F_0^j F_0^k \left[\sin\left((\omega_j-\omega_k)t\right) + \sin\left((\omega_j+\omega_k)t\right)\right]\sin(n_j z)\cos(n_k z). \end{aligned}\tag{20}$$

Regardless of the coincidence in indices j and k, averaging expression (20) in time gives zero. Thus in the case of elastic modes with vertically directed wave vectors, mass sources of non-linear origin do not produce long gravitational waves.

Now, let us estimate the contribution of non-linear effects, while taking into account acoustic modes that have wave vectors with horizontal components (propagating modes). In this case, the compressible fluid layer is infinite along the horizontal axis Ox, whereas the layer is bounded above and below by a free surface and absolutely rigid bottom, respectively. The elementary theory of wave guides [Brekhovskikh and Goncharov, 1994] points to elastic oscillations of a semi-bounded domain having a continuous frequency spectrum. Thus any elastic motion can be expressed as a superposition of normal waves

$$F(x,z,t) = \sum_j \int d\omega\, F_j^0(\omega)\cos(\omega t - m_j x)\,\sin(n_j z), \qquad (21)$$

$$m_j = \left(\omega^2 / c^2 - n_j^{\,2}\right)^{1/2}, \qquad (22)$$

where m_j is the horizontal component of the wave vector, and n_j is its vertical component (equation (15)). According to (22), only a finite number of modes of a fixed frequency ω can propagate in horizontal direction. These are called the “propagating modes”. For certain modes m_j is imaginary. Such modes decrease exponentially and thus cannot propagate along the axis Ox.

Expressing functions u', w', p' и ρ' in terms of the velocity potential (21), we obtain

$$\begin{aligned} f_x = \sum_{jk} \int d\omega_1 \int d\omega_2\, & F_j^0(\omega_1) F_k^0(\omega_2)\, m_{2k} \times \\ \times\{ & m_{1j} m_{2k} \sin(\omega_1 t - m_{1j}x)\cos(\omega_2 t - m_{2k}x)\sin(n_j z)\sin(n_k z) - \\ - & n_j n_k \cos(\omega_1 t - m_{1j}x)\sin(\omega_2 t - m_{2k}x)\cos(n_j z)\cos(n_k z) - \\ - & c^{-2}\omega_1\omega_2 \sin(\omega_1 t - m_{1j}x)\cos(\omega_2 t - m_{2k}x)\sin(n_j z)\sin(n_k z)\}, \end{aligned} \qquad (23)$$

$$\begin{aligned} f_z = \sum_{jk} \int d\omega_1 \int d\omega_2\, & F_j^0(\omega_1) F_k^0(\omega_2)\, n_k \times \\ \times\{ & -m_{1j} m_{2k} \sin(\omega_1 t - m_{1j}x)\sin(\omega_2 t - m_{2k}x)\sin(n_j z)\cos(n_k z) + \\ + & n_j n_k \cos(\omega_1 t - m_{1j}x)\cos(\omega_2 t - m_{2k}x)\cos(n_j z)\sin(n_k z) + \\ + & c^{-2}\omega_1\omega_2 \sin(\omega_1 t - m_{1j}x)\sin(\omega_2 t - m_{2k}x)\sin(n_j z)\cos(n_k z)\}, \end{aligned} \qquad (24)$$

$$\begin{aligned} s = \frac{1}{c^2}\sum_{jk} \int d\omega_1 \int d\omega_2\, & F_j^0(\omega_1) F_k^0(\omega_2)\, \omega_1 \times \\ \times\{ & m_{1j} m_{2k} \cos(\omega_1 t - m_{1j}x)\sin(\omega_2 t - m_{2k}x)\sin(n_j z)\sin(n_k z) + \\ + & m_{2k}^{\;2} \sin(\omega_1 t - m_{1j}x)\cos(\omega_2 t - m_{2k}x)\sin(n_j z)\sin(n_k z) - \\ - & n_j n_k \sin(\omega_1 t - m_{1j}x)\cos(\omega_2 t - m_{2k}x)\cos(n_j z)\cos(n_k z) + \\ + & n_j^{\;2} \sin(\omega_1 t - m_j x)\cos(\omega_2 t - m_{2k}x)\sin(n_j z)\sin(n_k z)\}. \end{aligned} \qquad (25)$$

It is important to stress here that elastic waves, with frequencies smaller than a minimum critical frequency do not exist, i.e. for $\omega_1 > \omega_0$ and $\omega_2 > \omega_0$. Replacing in (23)-(25) the products of "sin" and "cos" with sums of trigonometric functions, we get terms that oscillate with frequencies $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$. While averaging expression (25) in time, the terms with frequencies $\omega_1 + \omega_2$ vanish. At the same time, the terms with frequencies $\omega_1 - \omega_2$

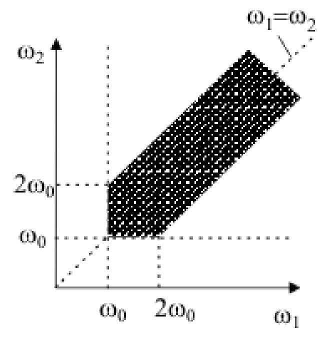

Figure 2. The domain of integration.

provide a nonzero contribution, but only if $\left|\omega_1 - \omega_2\right| < \omega_0$. Thus the domain of integration (Fig. 2) is a narrow band about the line $\omega_1 = \omega_2$.

According to (13), the following three values contribute to the generation of long gravitational waves:

$$"X" = \int_{-H}^{0} \frac{\partial f_x}{\partial x} dz, \qquad "Z" = \int_{-H}^{0} dz \int_{z}^{0} dz^* \frac{\partial^2 f_z}{\partial x^2},$$

$$"S" = \int_{-H}^{0} \frac{\partial s}{\partial t} dz.$$

Omitting intermediate steps, we get the final expressions:

$$"X" = \frac{H}{8c^4} \sum_j \int d\omega_1 \int d\omega_2 \; F_j^0(\omega_1) F_j^0(\omega_2) \cos\left[(\omega_1 - \omega_2)t - (m_{1j} - m_{2j})x\right] \omega \Delta\omega^3 \frac{n^2}{m^2}, \tag{26}$$

$$\begin{aligned} "Z" = {} & \frac{H}{4c^4} \sum_j \int d\omega_1 \int d\omega_2 \; F_j^0(\omega_1) F_j^0(\omega_2) \cos\left[(\omega_1 - \omega_2)t - (m_{1j} - m_{2j})x\right] \omega^2 \Delta\omega^2 \frac{n^2}{m^2} + \\ & + \frac{H}{8} \sum_{j \neq k} \int d\omega_1 \int d\omega_2 \; F_j^0(\omega_1) F_k^0(\omega_2) (m_{1j} - m_{2k})^2 \cos\left[(\omega_1 - \omega_2)t - (m_{1j} - m_{2k})x\right] \times \\ & \times \left\{ - m_{1j} m_{2k} \left(\frac{1+2k}{j-k} + \frac{1+2k}{1+j+k} \right) - n_j n_k \left(\frac{1+2k}{j-k} - \frac{1+2k}{1+j+k} \right) + \frac{\omega_1 \omega_2}{c^2} \left(\frac{1+2k}{j-k} + \frac{1+2k}{1+j+k} \right) \right\}, \end{aligned} \tag{27}$$

$$"S" = -\frac{H}{4c^4} \sum_j \int d\omega_1 \int d\omega_2 \; F_j^0(\omega_1) F_k^0(\omega_2) \cos\left[(\omega_1 - \omega_2)t - (m_{1j} - m_{2k})x\right] \omega^2 \Delta\omega^2. \tag{28}$$

When deriving the expressions (27)-(29), the following formulas and assumptions were used:

$$\int_{-H}^{0} \cos\left[\frac{\pi\beta}{H} z\right] dz = H \frac{\sin[\pi\beta]}{\pi\beta} = \begin{cases} H, \beta = 0 \\ 0, \beta = \pm 1, \pm 2, \pm 3, ... \end{cases},$$

$$\int_{-H}^{0} dz \int_{z}^{0} dz^* \sin\left[\frac{\pi\beta}{H} z^*\right] = \frac{H^2}{\pi^2 \beta^2} \left[\sin(\pi\beta) - \pi\beta\right] = \begin{cases} 0, \beta = 0 \\ -H^2 \pi^{-1} \beta^{-1}, \beta = \pm 1, \pm 2, \pm 3, ... \end{cases},$$

$\beta = j - k$ and $\beta = 1 + j + k$, $\Delta m = m_1 - m_2$, $\Delta\omega = \omega_1 - \omega_2$, $\Delta m \approx \frac{\omega}{c^2 m} \Delta\omega - \frac{1}{2\,c^2 m} \frac{n^2}{m^2} \Delta\omega^2$

which is a series expansion of (22) subject to condition $\Delta\omega << \omega$.

Expressions (26)-(28) allow us to conclude that the vertical component of the force f_z always dominates in the generation of long waves, whereas the contributions of the horizontal component f_x and the distributed mass source **s** are negligible ($\text{“Z”}/\text{"X"} \sim \omega/\Delta\omega >> 1$, $\text{"Z"}/\text{"S"} \sim n^2/m^2 >> 1$). In addition, expressions (26)-(28) testify to "non-linear tsunami source" propagation in horizontal direction with group velocity of an acoustic mode.

RESULTS AND DISCUSSION

The conclusions reached from analytical assumptions in the previous section are also confirmed by numerical calculations. Fig. 3 shows spatial distribution of mass force $\vec{f}$, calculated for three consecutive moments of time. It is seen that $\vec{f}$ is directed vertcally downwards almost everywhere.

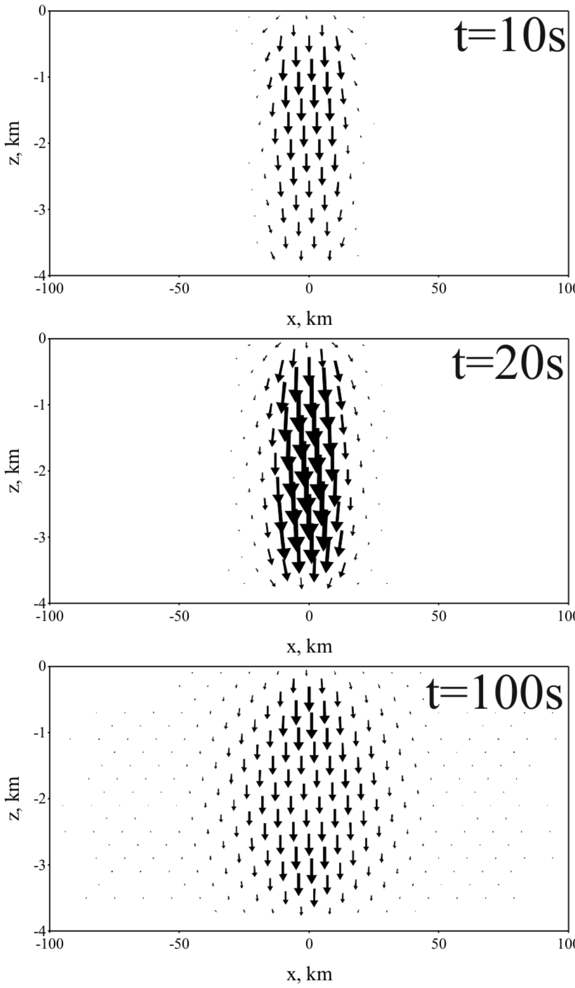

Figure 3. Evolution of the mass force field $\vec{f}$. Calculations were carried out for “piston” bottom displacement with τ=8s, L=50km, H=4km.

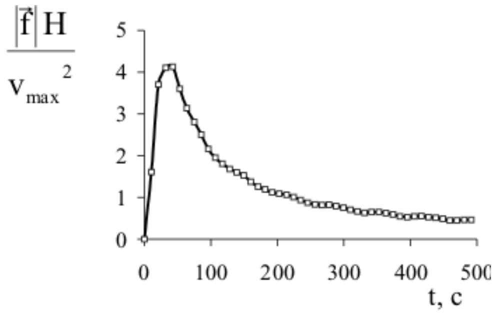

Figure 4. Time history of the absolute value of the mass force $\vec{f}$ in the centre of source area at a depth z=2 km. Calculations were carried out for a “piston” bottom displacement with τ=8s, L=50km, H=4km.

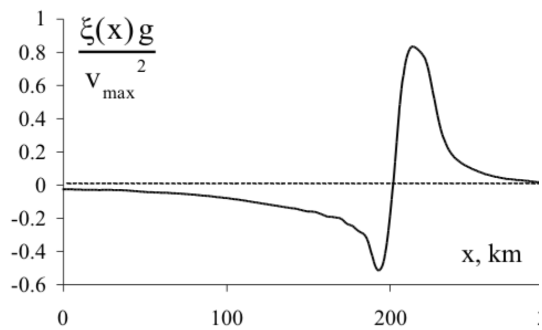

Figure 5. Gravitational wave generated by non-linear tsunami source at time t=400 H/c. Calculations were performed for a “piston” bottom displacement with τ=8s, L=50km, H=4km.

We get an idea about the absolute value of the force on Fig. 4, which shows a time-history of the module $\vec{f}$ on the centre of the active area at a depth of 2 km. As it follows from this figure, intensity of "non-linear tsunami source" reaches a maximum after a finite time, and not immediately. Therefore, as elastic waves leave the source area, the intensity of the non-linear tsunami source decreases.

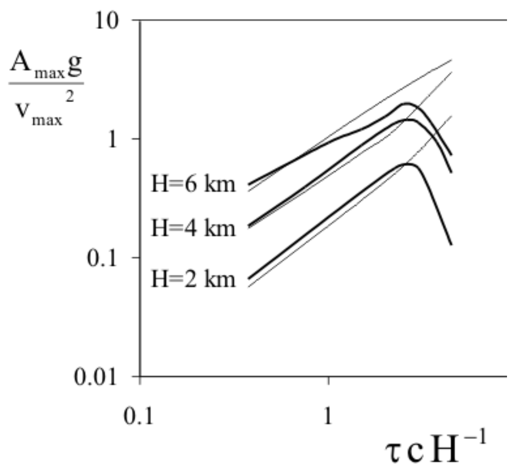

Figure 6. Amplitude of gravitational wave generated by non-linear tsunami source as a function of duration of the bottom displacement. Calculations were done for several ocean depths, H=2, 4, and 6 km, and source length of L=50 km. Thick line marks the "piston" bottom displacement, thin line – the "membrane" bottom displacement.

A typical gravitational wave, formed by a non-linear source, is presented on Fig. 5. This wave was calculated from a solution of equation (12). The wave consists of a positive leading crest and a rather long negative tail. The amplitude of the wave is given by v_{max}^2/g, where v_{max} is the maximum velocity of the bottom motion. It is easy to estimate that in the case of the maximum bottom movement velocity of ~1 m/s, the non-linear effect generates long wave with an amplitude of ~0.1m. This is significant amplitude for a tsunami in an open ocean.

The maximum wave amplitude generated by non-linear mechanism is plotted on Fig. 6 as a function of the duration of bottom displacement. This dependence is calculated for three different ocean depths in the cases of "piston" and "membrane" bottom motions. Because of the non-linear (acoustic) effects, increasing the ocean depth leads to larger wave amplitudes. The "membrane" bottom movements can generate waves of greater amplitude than the "piston" ones.

In conclusion, it is important to note that under "piston" bottom movements the non-linear mechanism can lead to observable, though not dominant contribution to the tsunami amplitude. However, the non-linear effects can play a dominant role during quick bottom motions without residual displacements, when the traditional linear mechanism is not able to effectively generate gravitational waves.

ACKNOWLEDGEMENTS

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